Recitation 3

September 10

Problem 1. Are the following collections of vectors linearly independent?

- $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$
- $\begin{bmatrix} 3 \end{bmatrix} \quad \begin{bmatrix} -1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Problem 2. Suppose you have a 5×7 matrix. When can its columns be linearly independent? How many pivot columns should the matrix have for its columns to span $\mathbb{R}^5$?

Problem 3. State the definition of a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

Problem 4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map sending a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the vector $\begin{bmatrix} x_1 + x_2 \\ x_2 - x_1 \end{bmatrix}$. Is T a linear transformation?

Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map sending a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the vector $\begin{bmatrix} x_1 + x_2 \\ x_2 - x_1 + 2 \end{bmatrix}$. Is F a linear transformation?

What about $Z\colon \mathbb{R}^2 \to \mathbb{R}^2$ sending $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

What about $S\colon \mathbb{R}^2 \to \mathbb{R}$ sending $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} x_1 x_2 \end{bmatrix}$?

Or $Q\colon \mathbb{R} \to \mathbb{R}$ sending $\begin{bmatrix} x_1 \end{bmatrix}$ to $\begin{bmatrix} 3x_1 \end{bmatrix}$?

Problem 5. For any matrix A one can define linear transformation $T(x) = Ax$. For each of the following matrices, determine dimensions of domain and codomain of corresponding transformations, and determine if each of the transformations is **onto**, **one-to-one**, or both.

- $\begin{bmatrix} -2 & 3 \\ 3 & 2 \end{bmatrix}$
- $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & -5 & -7 \\ -3 & 13 & 5 \end{bmatrix}$
- $\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$

- $\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$
- $\begin{bmatrix} -1 & 3 & 1 \end{bmatrix}$

For each of the non-one-to-one transformations T defined by matrices above, find all vectors x that are killed by T, i.e. such that $T(x) = 0$.

Problem 6. Find standard matrix of the transformation T sending

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} x_2 - 3x_3 \\ -4x_1 \\ 0 \\ 2x_3 - x_1 - x_2 \end{bmatrix}$$

What are the domain and codomain of that transformation?

Problem 7. Suppose you know that a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ sends vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and it sends vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Find the standard matrix of T using just the definition of linear transformation.

Problem 8. Find the standard matrix of the transformation $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates the plane counter-clockwise by the angle $\pi/4$.

Problem 9. Find the standard matrix of the transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ which reflects vectors with respect to y-axis.

Problem 10. Define a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which first rotates the plane counter-clockwise by the angle $\pi/4$, and then reflects it with respect to the y-axis. Find its standard matrix. Compare it to the product of matrices of the two previous exercises. Does the order of the product matter?

Problem 11. Define a transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ which sends each vector $v \in \mathbb{R}^3$ to its mirror image with respect to the xy-plane, and then rotates it around x-axis counter-clockwise by the angle $\pi/3$. Prove that it is indeed a linear transformation. Find the standard matrix of this transformation.